Tracking rapidly changing dynamical systems using a non-parametric statistical method based on wavelets

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SUMMARY
A non-parametric statistical method of tracking rapidly changing dynamical systems is introduced. The method implements the non-linear wavelet thresholding estimator to estimate the evolutionary transfer function of the system. The consistency and optimality of the resulting estimate of our method on rapidly changing systems are stated, while no linear estimator can achieve the same optimality. Two examples of real data are studied using this non-parametric method, including two vertical seismic array case studies and a series of seismic slope experiments. The analysis results are consistent with previous research for the vertical seismic array data. Moreover, our estimate is superior to those from previous research in the sense that our estimate is neither over-smoothed nor under-smoothed. The analysis results are also consistent with the experimental observations for the seismic slope experimental data. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: system identification; time-varying system; non-parametric statistics; wavelets; earthquake engineering; soil dynamics

1. INTRODUCTION
The ability to track a time-varying dynamical system would be important for earthquake engineers. With the knowledge of evolutionary dynamical properties, it is possible to understand the status and the underlying physical processes of the system. For instance, researchers in the area of structural health monitoring [1–4] are interested in knowing the changes of the mechanical properties of a structure during earthquakes. Methods based on wavelets have been also pursued [5, 6]. Also, geotechnical engineers are concerned with the responses of seismically-excited soils [7–11], whose mechanical properties also vary during an earthquake. Problems of tracking time-varying systems have usually been studied with adaptive linear methods, in which unknown parameters of the systems are estimated adaptively using

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local linear methods, and the desired physical characteristics evaluated using the estimated parameters. Examples include methods based on least squares [11–14], those based on the extended Kalman filter [9, 15], and non-parametric methods [7, 16], etc. An inherent assumption behind these linear methods is that the degree of change of the time-varying system is uniform through time [17]. This uniform-changing assumption is necessary for linear methods to behave well. Indeed, if the parameters of a linear system are allowed to vary wildly with time, the number of unknown parameters will quickly exceed the number of data, and the problem will become ill-posed. This renders linear methods unable to track systems with non-uniform degrees of change. We refer to such systems as rapidly changing systems.

On the other hand, estimating a rapidly changing function from its noisy observations has been made possible through the non-linear wavelet thresholding (NWT) estimator initially developed by Donoho and Johnstone in the early nineties [17–20]. The reasons for their success include non-linearity and the sparse representation under wavelet bases. The NWT estimator has offered a new possibility to track rapidly changing systems.

The purpose of this paper is to connect the problem of tracking rapidly changing systems to that of estimating rapidly changing functions, i.e. we take advantage of the adaptive properties of the NWT estimator to track rapidly changing physical systems. The resulting method of analyzing rapidly changing systems is a non-parametric model based on the NWT estimator. First, a time-varying system is modeled by a non-parametric model, then we introduce the NWT estimator and explain how we can combine the estimator and the non-parametric model to track rapidly changing systems. Two examples of real data will be analyzed at the end of this paper. The analyses of liquefied soil and slope failure, both due to earthquake excitations, show the wavelet non-parametric method is able to track rapidly changing systems.

2. RAPIDLY AND UNIFORMLY CHANGING FUNCTIONS

We refer to continuous-time functions whose regularities are uniform through time as ‘uniformly changing’ functions and those whose regularities are non-uniform through time as ‘rapidly changing’ functions. An example of uniformly changing functions is a Brownian motion (Figure 1(a)) whose regularity is constant through time [21]. Many realistic functions are rapidly changing functions. An example is a step function (Figure 1(b)), which is analytic everywhere but is discontinuous at the jump. This non-uniform changing behavior makes linear estimators inappropriate since linear estimators are only appropriate for uniformly changing functions.

To elaborate the above statement, consider the following example problem: given the noise-contaminated step function in Figure 2(a), the goal is to estimate the underlying step function, pretending that it is unknown. Figure 2(b) shows the estimate of a Butterworth low-pass filter (a linear estimator operating in the Fourier domain), and Figure 2(c) shows the estimate of the NWT estimator introduced in Section 4 using the Daubechies wavelets of order 1. The cut-off frequency of the Butterworth filter is chosen such that the degrees of oscillation in the smooth region of the estimates made by the two methods are roughly the same. In order to reduce the degree of oscillation in the smooth region, the low-pass filter inevitably over-smooths the sharp jump, as we can see in Figure 2(b). This is because the linear estimator assumes that the underlying function is uniformly changing; therefore, it cannot be simultaneously optimal.
Figure 1. (a) A Brownian motion. (b) A step function.

Figure 2. (a) The contaminated step function. (b) The estimate by a Butterworth low-pass filter. (c) The estimate made by the NWT estimator.
for a smooth curve and a sharp jump. On the other hand, the NWT estimator does not have this limitation.

Tracking a rapidly changing system is similar to estimating a rapidly changing function. When adaptive linear methods are used, the rapidly changing feature of the system can be lost. Therefore, the non-linear wavelet estimator is preferable in this case. In Sections 3 and 4, we discuss wavelet bases and the non-linear wavelet estimator, and in Section 5, we show the equivalence between tracking rapidly changing systems and estimating rapidly changing functions.

3. ORTHONORMAL WAVELET BASES

An orthonormal wavelet basis [21–23] spans the $L^2[0,1]$ space. Since the $L^2[0,1]$ space is infinite-dimensional, a wavelet basis contains infinitely many basis functions, called mother wavelets [21]. All mother wavelets have the same appearance, except that their scales and spatial (temporal) locations are different. Any mother wavelet can be written as a scaled and shifted version of a single function, called a wavelet function $\psi(t)$, i.e.

$$\psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k)$$

where $\psi_{j,k}$ is a mother wavelet indexed by integers $j$ and $k$, and the term $2^{j/2}$ is used to keep the $L^2$ norm (length) of a mother wavelet constant. Note that $\psi_{j,k}$ is defined on $[0,1]$; therefore, the index $k$ ranges from 1 to $2^j$. A wavelet function is a function that is transient in space or time, and so are mother wavelets. According to Equation (1), it is clear that the indices $j$ and $k$ control the scale (or duration) and location of $\psi_{j,k}$, respectively. A mother wavelet with large $j$ is a function with a short duration, and vice versa.

There is an interesting feature for an orthonormal wavelet basis: $L^2[0,1]$ is divided into orthogonal subspaces $W_i$, $i = -\infty, \ldots, \infty$ by mother wavelets [23]:

$$L^2[0,1] = W_{\infty} \oplus \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{\infty}$$

where $\oplus$ is the direct sum of linear spaces, and $W_j$ is the subspace spanned by $\{\psi_{j,k} : k = 1, \ldots, 2^j\}$, i.e. $W_j$ is the subspace spanned by all mother wavelets with scale $j$. This decomposition is called a multi-resolution analysis [22]. Now let

$$V_m = \bigoplus_{i=-\infty}^{m-1} W_i$$

then

$$L^2[0,1] = V_m \oplus W_m \oplus W_{m+1} \oplus \cdots \oplus W_{\infty}$$

Since $\psi_{j,k}$ with a large (or small) $j$ is a mother wavelet with a short (or long) duration, the projection of a function onto $W_j$ contains the detail (or gross) information of that function. On the other hand, the projection operation onto $V_m$ is a low-pass filter, with a frequency band equal to the union of the pass-bands of $W_{-\infty}, \ldots, W_{m-2}$, and $W_{m-1}$. This justifies the following interpretation to the partition of Equation (4): the projection of a function onto $V_m$ contains the trend of that function at scale $m$, while the projection onto $W_j$ ($j \geq m$) contains the oscillations at scale $j$. 

There exists an orthonormal basis of the subspace $V_m$. This basis contains the so-called father wavelets $\varphi_{m,k}$, $k = 1, \ldots, 2^m$ [23]. Father wavelets are indexed by a location parameter $k$. As a result, the multi-resolution analysis of a function $f(t)$ is as the following:

$$f = f^m_m + \sum_{j=m}^{\infty} f^j_j = \sum_{k=1}^{2^m} \langle f, \varphi_{m,k} \rangle \cdot \varphi_{m,k} + \sum_{j=m}^{\infty} \sum_{k=2}^{2^j} \langle f, \psi_{j,k} \rangle \cdot \psi_{j,k}$$

$$\equiv \sum_{k=1}^{2^m} \alpha_{m,k}(f) \cdot \varphi_{m,k} + \sum_{j=m}^{\infty} \sum_{k=1}^{2^j} \beta_{j,k}(f) \cdot \psi_{j,k}$$

(5)

where $f^m_m$ and $f^j_j$ are the projections of $f(t)$ onto $V_m$ and $W_j$, respectively, and contain the trend and oscillations of $f(t)$ at scale $m$ and $j$. The symbol $\langle \cdot, \cdot \rangle$ is the $L^2[0,1]$ inner product.

We call $\alpha_{m,k}(f)$ and $\beta_{j,k}(f)$ the wavelet coefficients of $f(t)$.

For a sampled function, we cannot compute its wavelet coefficients of all scales. Let the sample number $T$ be $2^J$, the most detailed subspace that we can consider is $V_J$. The approximate decomposition of $f$ is

$$f = \sum_{k=1}^{2^m} \alpha_{m,k}(f) \cdot \varphi_{m,k} + \sum_{j=m}^{J-1} \sum_{k=1}^{2^j} \beta_{j,k}(f) \cdot \psi_{j,k}$$

(6)

Note that, for a function of sample size $T = 2^J$, the total number of its wavelet coefficients is also $2^J$.

The computation of wavelet coefficients through a quadratic mirror filter algorithm (called the Fast Wavelet Transform) is faster than the Fast Fourier Transform [21]. Readers interested in the computational aspects of the wavelet transform are referred to Reference [21]. Wavelets that implement the quadratic mirror filter algorithm include the Daubechies wavelets, the Battle–Lemarie wavelets, the Coiflet wavelets, etc. Among them, the Daubechies wavelets [24] have compact regions of support. Since the compactly-supported properties are helpful for sparse representations of functions, in this paper we employ the Daubechies wavelets.

An appealing feature of wavelet bases is that many functions have sparse representations under wavelet bases, i.e. we only need a few father and mother wavelets to accurately represent most functions. The sparseness is due to the following two facts: (i) mother wavelets are orthogonal to some polynomials, and (ii) they have compact support (or nearly compact support) regions. A mother wavelet is said to have $r$ vanishing moments [21] if it is orthogonal to polynomials of degree less than $r$. Consequently, the wavelet coefficients $\beta_{j,k}$ of a polynomial of degree less than $r$ are all zeros. On the other hand, a function whose region of support does not intersect with the support region of a mother wavelet is orthogonal to that mother wavelet; hence, a small support region also helps the sparseness.

A smooth function, which is close to a polynomial, usually has a sparse representation under a wavelet basis; an irregular function, which usually contains sharp features having small support regions, also often has a sparse representation. Note that many bases of $L^2[0,1]$ do not have this property. For instance, an irregular function does not have a sparse representation under the Fourier basis since it requires infinitely high frequency components to represent the sharp edges.
Note that noise does not have sparse representations under any orthonormal basis. One such example is white Gaussian noise, which remains white Gaussian noise under any orthonormal basis. Therefore, the energy distributions of most signals, including smooth and rapidly changing ones, and noise are very different in the wavelet domain. In other domains, e.g. the Fourier domains, the energy distributions of most signals and noise are not always different, especially for rapidly changing functions whose frequency-domain representations are not sparse. So it is relatively easier to separate noise from signals in the wavelet domain.

4. NON-LINEAR WAVELET THRESHOLDING ESTIMATOR

The setting of the NWT estimator starts with the following equation:

$$y(i/T) = f(i/T) + z(i/T) \quad i = 1, 2, \ldots, T$$  \hspace{1cm} (7)

where $y$, $f$, and $z$ are observed data, the unknown underlying function, and Gaussian noise, respectively. The goal is to estimate $f$ based on $y$, i.e. to separate $z$ from $y$. The estimator was initially proposed by Donoho and Johnstone in a series of papers [17–20]. They showed that the estimator is superior to linear estimators in the sense that if $f$ contains rapidly changing features, the procedure is able to efficiently remove the noise $z$ without destroying these features; nevertheless, no linear estimator can effectively achieve this. This desirable property of the NWT estimator was named spatial adaptation [25]. The key of the success is twofold: (i) wavelet bases provide sparse representations for most functions (i.e. function $f$) compared to other bases, and (ii) the approach relies on a non-linear estimator in the wavelet domain.

Owing to the sparse representation by wavelets, most coefficients of $f$ are close to zero, and those not close to zero are large (due to energy conservation, a consequence of the Parseval theorem). In contrast, most wavelet coefficients of $z$ are not zero, so they are typically quite small. This leads to the following simple de-noising procedure: given a wavelet coefficient of $y$, denoted by $\alpha(y)_{m,k}$ or $\beta(y)_{j,k}$, we need to decide if it is mostly due to noise or not, and if so, we should discard the coefficient by setting it to zero. A natural way of making this decision is to select a threshold level, denoted by $\lambda_{j,k}$, which should reflect the noise amplitude. If $|\beta(y)_{j,k}|$ is larger than $\lambda_{j,k}$, we believe it contains information about $f$, and vice versa. As a result, the problem of estimating $f$ reduces to one of choosing appropriate threshold levels $\lambda_{j,k}$ for $\beta(y)_{j,k}$. Notice that we usually do not threshold $\alpha(y)_{m,k}$ since they mostly contain information about the gross structure of $f$ and are already much larger than noise level [17].

An appropriate choice of threshold level relies on the knowledge of the structure of noise $z$. We will be exclusively concerned with the situation where $z$ is stationary Gaussian. In the case that $z$ is non-Gaussian, suitable estimation procedures are discussed in Donoho and Yu [26] and will not be pursued in this paper. For the Gaussian case, the NWT estimator is as follows [27]:

(i) Compute the wavelet coefficients of $y$, i.e. $\alpha(y)_{m,k}$ and $\beta(y)_{j,k}$.

(ii) Estimate the standard deviation of $\beta(y)_{j,k}$, denoted by $\hat{\sigma}_j$, by the median value of $\{|\beta(y)_{j,k}| \quad k = 1, \ldots, 2^j\}$ divided by 0.6745 (specific for Gaussian noise) and compute
the threshold level as
\[ \hat{\lambda}_{j,k} = \delta_j \sqrt{2 \log(n)} \] (8)

(iii) Threshold \( \beta(y)_{j,k} \)
\[ \hat{\beta}(y)_{j,k} = \begin{cases} \beta(y)_{j,k} & |\beta(y)_{j,k}| > \hat{\lambda}_{j,k} \\ 0 & \text{otherwise} \end{cases} \quad j = m, \ldots, J - 1 \quad k = 1, \ldots, 2^j \] (9)

(iv) Find the estimate of \( f \), denoted by \( \hat{f} \), by the inverse wavelet transform
\[ \hat{f} = \sum_{k=1}^{2^n} a(y)_{m,k} \varphi_{m,k} + \sum_{j=m}^{J-1} \sum_{k=1}^{2^j} \hat{\beta}(y)_{j,k} \varphi_{j,k} \] (10)

Equation (9) shows that \( \hat{\beta}(y)_{j,k} \) depends on \( \beta(y)_{j,k} \) in a non-linear way, i.e. \( \hat{f} \) is a non-linear estimator of \( f \). This non-linearity is necessary for preserving rapidly changing features of \( f \). The estimator is nearly minimax over two function scales, the Besov and Triebel function scales [28], while no linear estimator, e.g. a kernel method or a low-pass filter, can be nearly-minimax over the two scales [17].

5. TRACKING RAPIDLY CHANGING SYSTEMS

The problem of tracking a rapidly changing system is defined as follows. Given the input and output of an unknown rapidly changing system, we are interested in estimating the evolution of its transfer function through time, which we will call the ‘evolutionary transfer function’ (ETF). In this paper, we assume that the rapidly changing system can be modeled by the following model:
\[ Y(i/T) = \sum_{p=0}^{r} a_p(i/T) \cdot X[(i - p)/T] + e(i/T) \quad i = 1, 2, \ldots, T \] (11)

where \( T \) is the number of data points, \( r + 1 \) is the number of filter coefficients used at each time instant, \( Y(i/T) \) and \( X(i/T) \) are the output and input of the unknown system at time \( i/T \), \( e(i/T) \) is noise, and \( a_p(i/T) (i = 1, \ldots, T; \ p = 0, \ldots, r) \) are rapidly changing functions that characterize the ETF of the unknown system. The ETF is defined by \( B(i/T, \omega) \), where
\[ B(i/T, \omega) = \sum_{p=0}^{r} a_p(i/T) e^{-j\omega p} \] (12)

Note that we have assumed in Equation (11) that all functions, \( X, Y, a_p, \) and \( e \), are defined on the interval [0,1]. We will discuss why we do so in Appendix A. Given \( X \) and \( Y \), our goal is to estimate the \( r + 1 \) functions, \( a_p (p = 0, \ldots, r) \), presumably changing rapidly with time. The above model uses \( r + 1 \) parameters to characterize the system at each time instant; for a fixed time Equation (11) is in fact a finite-impulse-response filter equation, and the time evolution of \( a_p \) is non-parametric.

Notice that if \( r > 0 \), the model is ill-posed since the total number of unknowns is \( (r + 1)T \), greater than the number of data points \( T \). This difficulty is the principle reason why linear
methods must take the uniformly changing assumption. With the assumption, the problem may become well-posed, but the assumption also rejects the possibility of a rapidly changing system. Instead, we assume that \( a_p \) is in a certain function class, called the Hölder function class of regularity one, denoted by \( H^1[0,1] \) [23]. Any function \( f \) on \([0,1]\) satisfying the following condition is in this class:

\[
H^1[0,1] = \{ f : |f(t_1) - f(t_2)|/|t_1 - t_2| < \infty, \ \forall t_1, t_2 \in [0,1] \}
\]  

(13)

\( H^1[0,1] \) contains uniformly changing as well as rapidly changing functions. Since \( H^1[0,1] \) belongs to the Besov scale, the NWT estimator is nearly minimax.

A common way of estimating \( a_p \) is to first estimate the ‘evolutionary power spectrum’ [29], see Appendix (A) of X and the ‘evolutionary cross spectrum’ between \( X \) and \( Y \), and \( a_p \) can be estimated using the estimated power and cross spectra. The evolutionary power and cross spectra are denoted by \( \hat{f}_{xx}(i/T, \omega) \) and \( \hat{f}_{xy}(i/T, \omega) \), and their estimates are denoted by \( \hat{\hat{f}}_{xx}(i/T, \omega) \) and \( \hat{\hat{f}}_{xy}(i/T, \omega) \),

\[
\hat{\hat{f}}_{xx}(i/T, \omega) = \frac{1}{N + 1} \left| \sum_{m=-N/2}^{i+N/2} h[(i - m)/T] \cdot X(m/T) \cdot e^{-j\omega n/(N+1)} \right|^2
\]

\[
\hat{\hat{f}}_{xy}(i/T, \omega) = \frac{1}{N + 1} \left( \sum_{m=-N/2}^{i+N/2} h[(i - m)/T] \cdot X(m/T) \cdot e^{-j\omega n/(N+1)} \right)^* \times \left( \sum_{m=-N/2}^{i+N/2} h[(i - m)/T] \cdot Y(m/T) \cdot e^{-j\omega n/(N+1)} \right)
\]

In Equation (14), \( h \) is a window function, e.g. the Hanning window, of width \( N + 1 \) used to reduce the leakage of the estimate [30]. The estimate of the evolutionary power (or cross) spectrum is actually a time-localized version of the estimate of the power (or cross) spectrum for stationary random processes. Note that Equation (14) has introduced a certain amount of smoothing in the temporal domain into \( \hat{f}_{xx} \) and \( \hat{f}_{xy} \), and the amount of the smoothing depends on the choice of \( N \). This smoothing is necessary since the information about \( f_{xx}(i/T, \omega) \) and \( f_{xy}(i/T, \omega) \) is contained not only in \( Y(i/T) \) and \( X(i/T) \) but also in the \( X \)s and \( Y \)s near to time instant \( i/T \). However, it is better not to over-smooth the estimates since the rapidly changing feature of the spectrum would be lost. This is another reason why a bell-shaped function centered at 0 is used as the window. Such a window function puts more weight on observations close to \( i/T \) when estimating \( f_{xx}(i/T, \omega) \) and \( f_{xy}(i/T, \omega) \).

The estimated \( a_p \), denoted by \( \hat{a}_p \), are

\[
\hat{a}_p(i/T) = \sum_{m=-N/2}^{N/2} \hat{f}_{xy}(i/T, \omega) \cdot e^{j\omega p/(N+1)} \quad i = 1, 2, \ldots, T \quad p = 0, 1, \ldots, r
\]

(15)

Since \( \hat{a}_p \) are noisy observations of \( a_p \), we assume the following equation holds,

\[
\hat{a}_p(i/T) = a_p(i/T) + e_p(i/T) \quad i = 1, 2, \ldots, T \quad p = 0, 1, \ldots, r
\]

(16)
In Appendix A, we will introduce some results proven by other researchers to show that \( \hat{a}_p \) is a consistent estimate of \( a_p \). However, \( \hat{a}_p \) is usually a ‘noisy observation’ of \( a_p \). In general, we cannot take \( \hat{a}_p \) as our final estimate of \( a_p \) since in many cases it is too noisy, and a de-noising procedure is required to get a sharper estimate of \( a_p \). The \( r + 1 \) equations of Equation (16) are the key connections between our original problem, the one of estimating the ETF, and the NWT estimator. \( \hat{a}_p \), \( a_p \), and \( e_p \) play the same roles as \( y \), \( f \), and \( z \) of Equation (7), respectively. Since the system changes rapidly, we expect the \( r + 1 \) functions \( a_p \) also to change rapidly through time, and the use of the NWT estimator is essential.

It is necessary to know the probabilistic structure of \( e_p \) to implement the NWT estimator. We assume \( e_p \) is a stationary Gaussian process, and this assumption comes from our experience with real data and is purely heuristic. Based on this assumption, the NWT estimator introduced previously can be used to estimate \( a_p \) given \( \hat{a}_p \). Denoting the estimation result as \( \tilde{a}_p \), we can estimate ETF of the system as

\[
\tilde{B}(i/T, \omega) = \sum_{p=0}^{r} \tilde{a}_p(i/T)e^{-j\omega p}
\]  

(17)

By inspecting \( \tilde{B} \), the evolutionary natural frequency and damping ratio of the system can be found by plotting out \( |\tilde{B}(i/T, \omega)| \), called evolutionary spectral amplitude, and identifying the peaks of \( |\tilde{B}(i/T, \omega)| \). The position of a peak indicates the value of the natural frequency, and the damping ratio can be estimated from the half-peak bandwidth.

6. EXAMPLES OF REAL DATA

We present two examples of real data analyses. The first example is from two vertical seismic array sites, including a liquefaction event, and the second example is from laboratory experiments on seismically-induced permanent slope deformations. Most of the cases are rapidly changing dynamical physical systems. In all cases, we implement the non-parametric model and the NWT estimator to compute \( \tilde{B} \), the estimated ETF of the soil systems. According to the patterns found from the estimated ETF, we make some comments about the possible physical processes. For validation, the comments will be compared with conclusions of relevant previous research and experimental observations.

Vertical seismic array data analysis

Vertical seismic arrays are full-scale in-situ testing facilities. This example focuses on the identification of soil degradation behaviors during seismic excitations using vertical seismic array data. We are exclusively concerned with downhole–uphole records from which earthquake source mechanisms and travel path effects can be isolated. Two sites are studied: the first site is the Lotung large-scale seismic test (LSST) site in Taiwan, and the second one is the Wildlife Refuge site in Imperial County, California. The two studied earthquakes are the LSST16 earthquake at the Lotung site and the Superstition Hills earthquake at the Wildlife site.

Lotung site. The Lotung site is located in north-eastern Taiwan. The depths of the downhole triaxial accelerometers are 0 m, 6 m, 11 m, 17 m, and 47 m [31]. In this study, only the 0 m
Table I. The general information of the analyzed events for the vertical seismic array case study.

<table>
<thead>
<tr>
<th>Event name</th>
<th>Date</th>
<th>Epic. dist. (km)</th>
<th>Local mag.</th>
<th>PGA (g)</th>
</tr>
</thead>
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<td>Lotung site</td>
<td>11/14/86</td>
<td>77.9</td>
<td>7.0</td>
<td>0.17</td>
</tr>
<tr>
<td>LSST16</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wildlife Refuge site</td>
<td>11/24/87</td>
<td>31</td>
<td>6.6</td>
<td>0.21</td>
</tr>
<tr>
<td>Superstition Hills</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3. (a) The LSST16 accelerations recorded at the ground surface, and (b) the LSST16 accelerations recorded at 6 m deep. (c) The 3D plot of the estimated ETF for the AC4–6 pair (the horizontal axis is time), (d) the rotated view of the same estimated ETF (the horizontal axis is frequency).

and 6 m records of the DHB array are analyzed. The local soil profile includes 30–34 m thick silty sand and sandy silt overlain clayey silt and silty clay to a depth of about 400 m [31]. The water table was within 1 m of the surface [31]. The properties of the LSST16 seismic event are listed in Table I. The acceleration time histories at depths of 0 m and 6 m below ground surface are shown in Figures 3(a) and (b). The histories at 6 m and 0 m are regarded as the input \( X \) and output \( Y \) of our model (Equation (11)), and our goal is to estimate the ETF of the soil layer between 0 m and 6 m. The Daubechies wavelet of order 8 is used in this analysis. The resulting 3D surface plots of \( |\tilde{B}(i/T, \omega)| \) through time are shown in Figures 3(c) and (d), where the two axes are time and frequency.
There is a clear spectral peak ridge ranging from 3.5 Hz to 6 Hz, indicated as the gray line in Figure 3(d), throughout the entire earthquake duration. This spectral ridge corresponds to the estimated evolutionary fundamental natural frequency of the 6 m thick soil layer, and the width of this peak is indicative of the estimated evolutionary damping ratio. Note that the estimated natural frequency of the soil decreases with increasing seismic motions, and the estimated damping ratio is high during the time period of strongest motions 20–25 s (during the time period the spectral peak has larger bandwidth). These results are compatible to our common understanding of soil degradation behavior. Also, this result is consistent with the results of Chang et al. [7], Elgamal et al. [8], Glaser and Baise [32], and Ching and Glaser [33]. All of these researchers adapted linear methods and obtained similar results about the degradation of the soil. For this case study, linear methods are good enough to track the slowly time-varying soil system, and our method also yields satisfactory results.

Wildlife Refuge site. The Wildlife Refuge site is located in Imperial County, California. The site consists of 2–3 m of sandy silt and clayey silt, underlain by 5 m of loose silty sand over stiff clayey silt and stiff silt [34]. Two accelerometers are installed at depths of 0 m and 7.5 m, together with five piezometers installed within the loose silt layer. On 24, November 1987, the Superstition Hills earthquakes struck the site, and the loose silty sand layer liquefied [35]. The north–south component of the acceleration, together with the pore-water pressure time history measured at the depth of 2.9 m, is shown in Figure 4.

Previous research about this event includes Zorapapel and Vucetic [36], Glaser [9], Zeghal et al. [10], etc. Among them, Glaser evaluated the evolutionary natural frequency (shown in Figure 5) using the extended Kalman filter, and Zeghal et al. evaluated the evolutionary shear wave velocity (shown in Figure 6) of the 7.5 m thick soil system using the windowed Fourier spectral ratio. Their estimation methods are both linear. Zeghal et al. further analyzed the derived stress–strain diagrams. They found that strain hardening at large
Figure 5. The evolutionary natural frequencies of the 7.5 m thick soil estimated by Glaser [9].

Figure 6. The evolutionary shear wave velocity of the 7.5 m thick soil estimated by Zeghal et al. [10].
strain levels is evident, and that the pronounced strain hardening correlates well with the recorded sharp pore-water pressure drops (indicated by point numbers 1–3, 5–8, and 10–12 in Figure 4), presumably due to hardening by soil dilation. However, these peculiar strain-hardening behaviors were not found in Glaser’s results (Figure 5). In Zeghal et al.’s result (Figure 6), each peculiar peak is associated with a jump in Figure 6, but there are many jumps in Figure 6 not associated with any peculiar peak. We believe that Glaser over-smoothed and Zeghal et al. under-smoothed the estimated evolutionary natural frequency or shear wave velocity, and these are the inevitable consequences of using linear methods.

Using the wavelet non-parametric method with the Daubechies wavelet of order 8, we also estimated the ETF. The acceleration histories at depths of 7.5 m and 0 m are regarded as the input $X$ and output $Y$ of our model. The estimated $|\tilde{B}(i/T, \omega)|$ is shown in Figure 7. In this figure, we see that during the first twelve seconds, the 7.5 m thick soil system exhibits two modes of about 4 Hz and 10 Hz. After 12 s, the second mode disappears, and the first mode declines dramatically to a small value, indicating the onset of the liquefaction. This low-frequency mode runs throughout the entire duration after 20 s, indicating the soil was

Figure 7. Top: the excess pore-water pressure history. Bottom: the estimated ETF of the 7.5 m thick soil system.
continuously liquefied after 20 s (see the gray line in Figure 7(bottom)). Moreover, at most of the previously mentioned peculiar points, isolated high-frequency peaks are found, i.e. around these peculiar points, some higher modes appear, due to the fact that the soil system hardens and softens rapidly. This result correlates well with the finding of Zeghal et al. from the stress–strain diagrams. At the time instants of the peculiar points 2, 5, 6, 7, 8, 11 and 12, prominent isolated peaks are found in the estimated ETF, indicating strong hardening. For the time period during which the peculiar points are absent, significant high-frequency peaks are seldom found. These results indicate that our method performs satisfactorily in tracking the rapidly changing 7.5 m thick soil system without over- and under-smoothing.

Based on Figure 7, we divide the entire duration of the event into four stages. They are (1) stage 1 (0–12 s): the two resonant modes are clear, indicating the soil degradation is not significant in this stage. (2) Stage 2 (12–20 s): the first natural frequency decreases rapidly, indicating the soil degrades rapidly. (3) Stage 3 (20–42 s): isolated high-frequency peaks occur, indicating rapidly changing soil dynamical behaviors. It is believed that these peaks are due to the strain hardening and dilation of the soil. (4) Stage 4 (42 s–end): high-frequency peaks are absent for most of the time, i.e. the soil cannot convey high-frequency energy, indicating complete liquefaction. Although the division of the four stages is completely based on Figure 7, our results are very similar to the way that Zeghal et al. divided the duration of this event (as seen in Figure 6) according to the calculated stress–strain diagrams and the recorded pore-water pressure data.

**Model slope experiment data analysis**

Estimating permanent displacements of a soft soil slope induced by seismic strong shaking is of great interest to geotechnical earthquake engineers. The well-known Newmark rigid block analysis was an early attempt at analyzing the slope displacement problem [37]. According to the analysis method, the permanent displacement of a slope is evaluated by double integrating the acceleration time-history segments during which the seismically-induced inertial force is greater than the shear resistance of the soil slip surface. Incremental advances have been made based on the Newmark analysis: ‘decoupled’ analyses [38, 39] were proposed to solve the slope displacement problem with an extra dynamical analysis step, in which shear stress and acceleration time histories in the soil are evaluated, before implementing the Newmark-type analysis. In the dynamical analysis step, the slope is assumed to be a continuum. On the other hand, ‘coupled’ analyses [40, 41] refer to those acknowledging the fact that the deforming slope is no longer continuous.

Although various slope displacement analyses have been proposed, well-documented full-scale seismic field tests are scarce. In view of this, a project was launched by the University of California at Berkeley [42], and six physical model slope experiments were conducted by Wartman [43]. The purpose of that project was (i) to investigate the mechanisms of seismically-induced permanent deformations in slopes and embankments, (ii) to assess the accuracy and applicability of the Newmark-type procedures for estimating deformations in slopes, and (iii) to develop a series of fully defined model-scale case histories for calibration of existing numerical procedures for predicting seismic slope deformations and for the future development of more advanced numerical analyses.

We will analyze the seismic slope displacement experimental data using the non-parametric model with the NWT estimator. The non-parametric model and thresholding estimator were
chosen because we expect the behaviors of the model slope systems to change rapidly with time, especially after the slopes start to slide. The analyzed experiments are the second, third, fourth, and sixth tests conducted by Wartman, denoted by experiment #2, #3, #4, and #6. The geometries of the model slopes used by the four experiments are shown in Figure 8. All model slopes were constructed in a rigid box container bolted to a shaking table and were composed of soft clay over stiff clay. The heights of the model slopes were 18.5, 21.8, 31.2, and 21.8 cm for the four experiments, respectively, and the lengths and widths are roughly 127 cm and 94 cm. The locations of the accelerometers and potentiometer/extensometers are indicated in Figure 8. In experiment #6, sand was added to the soft clay to increase the unit weight of the clay, and a low-friction geo-synthetic interface (location indicated in Figure 8(d)) was inserted as a preferential sliding plane between the soft and stiff clays. The properties of the soft and stiff clays used in each experiment are compiled in Table II.

The model slopes were excited by lateral strong motions, and accelerations and absolute displacements were measured during each experiment. The mean frequencies and maximum horizontal accelerations of these input motions are compiled in Table III. The model slope #3 and #6 were subject to two strong motions, roughly 15 and 60 min apart. We will refer, for instance, to the acceleration record at accelerometer #6 as AC6 and the displacement record at potentiometer/ extensometer #13 as DP13. Further details about the experiments can be found in Reference [43].

The four experiments are analyzed as follows: the acceleration data is examined as pairs, with the deeper acceleration data as the input to our non-parametric model and the shallower one as the output. The ETF is estimated for each data pair using the non-linear thresholding estimator, providing information of the evolution of the dynamical behaviors of the model slopes. Patterns present in the estimated ETF will be identified, and detailed discussions about the dynamical behaviors and sliding mechanisms for the model slopes given.

The estimated ETFs can be classified into two categories according to the locations of the input/output accelerometer pair: (i) the trans-sliding-surface cases—the input accelerometer is beneath the sliding surface, while the output accelerometer is above the sliding surface, and (ii) the cis-sliding-surface cases—both input and output accelerometers are above the sliding surface. The patterns present in the estimated ETFs for these two cases are usually quite different.

Experiments #2 and #3. Experiments #2, #3, and #4 are a series of three tests considering small, moderate, and large displacements along deep sliding planes. Experiments #2 and #3 yielded similar estimated transfer functions and are discussed together. Figure 9 shows the (pre-test and post-test) profiles of the model slope of experiment #3, and Figure 10 shows the recorded data at AC4, AC6 and DP14 (other displacement data show similar patterns), and the estimated evolutionary transfer function between the AC4–6 pair for experiment #3. Notice that this is a trans-sliding-surface case since AC4 is below the sliding plane, but AC6 is above the plane. In the estimated ETF for this event, there is a region, indicated in Figure 10(d), of nearly zero high-frequency spectral amplitude, which starts roughly at 6 s, gradually extends its size during 6–7 s, is fully developed during 7–12 s, and gradually shrinks after 12 s. The extent of the nearly-zero region is less clear after 12 s, as indicated by the dashed lines in Figure 10(d). Compared to the recorded displacement at DP14 (Figure 10(c)), the fully-developed nearly-zero region (7–12 s) coincides with the duration when the recorded permanent displacement of the slope is pronounced.
Figure 8. The geometries of the model slopes of experiments #2, #3, #4 and #6. The location of the geosynthetic for experiment #6 is indicated. The triangles and rectangles are the locations of the accelerometers and displacement potentiometers (or extensometers).
There are two observable resonance modes whose natural frequencies start from roughly 10 Hz and 18 Hz at 0 s, decline during 4–7 s, stay at roughly 5 Hz and 13 Hz during 7–12 s, and return to roughly 9 Hz and 16 Hz after 12 s. Note that at the end of the experiment, the natural frequencies do not completely return to their starting values. Using the $V_s/4H$ rule, the first mode implies that the shear wave velocity of the soft clay starts from 8.2 m/sec at 0 s, stays at 4.1 m/sec during 7–12 s, and returns to 7.9 m/sec after 12 s. The shear wave velocity of the soft clay evaluated by Wartman [43] using a small-strain test was 6.5 m/sec (as shown in Table II).

**Table II.** The properties of the soft and stiff clay used in the four experiments.

<table>
<thead>
<tr>
<th>Slope #2</th>
<th>Soft model clay</th>
<th>Stiff model clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water content (%)</td>
<td>128.0 ± 1.5</td>
<td>105.2 ± 0.4</td>
</tr>
<tr>
<td>Peak undrained shear strength (psf)</td>
<td>46.0 ± 3.0</td>
<td>86.0 ± 5.0</td>
</tr>
<tr>
<td>Residual undrained shear strength (psf)</td>
<td>33.0 ± 5.0</td>
<td>62.3 ± 3.0</td>
</tr>
<tr>
<td>Shear wave velocity (ft/sec)</td>
<td>21.1</td>
<td>66.9</td>
</tr>
<tr>
<td>Wet density (pcf)</td>
<td>86.5</td>
<td>86.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Slope #3</th>
<th>Soft model clay</th>
<th>Stiff model clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water content (%)</td>
<td>127.7 ± 1.6</td>
<td>122.8 ± 0.3</td>
</tr>
<tr>
<td>Peak undrained shear strength (psf)</td>
<td>49.0 ± 3.0</td>
<td>90.0 ± 5.0</td>
</tr>
<tr>
<td>Residual undrained shear strength (psf)</td>
<td>32.5 ± 5.0</td>
<td>57.0 ± 7.0</td>
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<tr>
<td>Shear wave velocity (ft/sec)</td>
<td>21.3</td>
<td>62.5</td>
</tr>
<tr>
<td>Wet density (pcf)</td>
<td>86.5</td>
<td>86.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Slope #4</th>
<th>Soft model clay</th>
<th>Stiff model clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water content (%)</td>
<td>128.1 ± 0.8</td>
<td>115.0 ± 1.9</td>
</tr>
<tr>
<td>Peak undrained shear strength (psf)</td>
<td>56.0 ± 3.0</td>
<td>124.0 ± 5.0</td>
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<tr>
<td>Residual undrained shear strength (psf)</td>
<td>37.0 ± 6.0</td>
<td>84.2 ± 6.0</td>
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<td>Shear wave velocity (ft/sec)</td>
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</tr>
<tr>
<td>Wet density (pcf)</td>
<td>86.5</td>
<td>86.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Slope #6</th>
<th>Soft model clay</th>
<th>Stiff model clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water content (%)</td>
<td>34.5 ± 0.3</td>
<td>110.0 ± 0.9</td>
</tr>
<tr>
<td>Peak undrained shear strength (psf)</td>
<td>33.9 ± 1.1</td>
<td>144.5 ± 2.0</td>
</tr>
<tr>
<td>Residual undrained shear strength (psf)</td>
<td>15.0 ± 2.0</td>
<td>58.2 ± 2.0</td>
</tr>
<tr>
<td>Shear wave velocity (ft/sec)</td>
<td>46.0</td>
<td>61.2</td>
</tr>
<tr>
<td>Wet density (pcf)</td>
<td>115.5</td>
<td>86.5</td>
</tr>
</tbody>
</table>

**Table III.** The properties of the input motions for the four experiments.

<table>
<thead>
<tr>
<th>Experiment number</th>
<th>Maximum horizontal acceleration (g)</th>
<th>Mean frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.0</td>
<td>21.0</td>
</tr>
<tr>
<td>3 (first input)</td>
<td>2.3</td>
<td>21.0</td>
</tr>
<tr>
<td>3 (second input)</td>
<td>2.3</td>
<td>21.0</td>
</tr>
<tr>
<td>4</td>
<td>3.46</td>
<td>13.0</td>
</tr>
<tr>
<td>6 (first input)</td>
<td>4.0</td>
<td>20.0</td>
</tr>
<tr>
<td>6 (second input)</td>
<td>0.49</td>
<td>6.0</td>
</tr>
</tbody>
</table>

Figure 11 shows the recorded data at AC4, AC7 (this is another trans-sliding-surface case) and DP14, and the estimated evolutionary transfer function between the AC4–7 pair of experiment #3. A smaller nearly zero region, whose position of the region is close to that of Figure 10(d), is observable. There are three clear modes, and the positions of the first two modes are close to the two modes observed for the AC4–6 pair. At the end of the experiment, the natural frequencies do not completely return to their starting values. Among the three modes, the second and third modes are interrupted several times during 7–12s. However, at two time instants, 8s and 10.5s, the two modes resume. These two time instants correlate well with the two time instants at which sliding temporarily stops or slows down (see Figure 11(c)).

For most trans-sliding-surface cases of experiments #2 and #3, i.e. the AC5–9, AC5–4, AC5–7 pairs of experiment #2 and the AC4–9 pair of experiment #3, similar patterns are found in the estimated evolutionary spectra. The only exception is the AC5–6 pair of experiment #2, for which the nearly zero region is not observed.

On the other hand, the estimated ETFs of the cis-sliding-surface cases show dramatically different patterns. Figures 12(d) and (e) show the estimated ETF for the AC6–7 pair of experiment #3. The figure shows no nearly zero region, but the declination of the natural frequency is still evident. The same results are found for other cis-sliding-surface cases, i.e. the AC7–4, AC7–6, and AC7–9 pairs of experiment #2 and the AC6–9 pair of experiment #3.

The patterns present in the estimated ETF for experiments #2 and #3 can be summarized as follows. (i) nearly zero regions—when significant permanent displacement occurs, the high-frequency spectral ratio of any trans-sliding-surface pair usually declines close to zero. The nearly zero regions grow gradually before their fully developed stages and also shrink slowly after the fully developed stages. No nearly zero region is observed for any cis-sliding-surface pair. (ii) Degradation—the natural frequencies of the soft clay, which can be identified by the spectral peaks of the estimated ETF of any across- or cis-sliding-surface case, reduces
Figure 10. Results of experiment #3: (a) the acceleration record at AC6; (b) the acceleration record at AC4; (c) the displacement at DP14; (d) the estimated ETF for the AC4–6 pair (the horizontal axis is time); and (e) the rotated view of the same estimated ETF (the horizontal axis is frequency).

with increasing shaking level. Moreover, the natural frequencies usually do not return to their original values after the strong shaking. (iii) Interruption of spectral peaks—in some trans-sliding-surface cases, the spectral peaks of higher natural frequencies are interrupted when significant sliding occurs. But the peaks may soon resume when sliding temporarily stops or slows down.

Experiments #4. Experiment #4 is the last in a series of three tests that consider small, moderate, and large displacements that occur along deep sliding planes. The slope displaced more than 12.7 cm laterally, which is the largest deformation of any of the model slopes. In general, the patterns in the estimated ETFs of experiments are slightly different from those of experiments #2 and #3: (i) for the trans-sliding-surface cases, the nearly-zero regions appear faster and extend wider than those of experiments #2 and #3. (ii) for the trans-sliding-surface cases, the resonance modes are not clear after sliding occurs.

Figure 13 shows the profiles of the model slope of experiment #4, and Figure 14 shows the recorded data at AC4, AC6 (trans-sliding-surface) and DP16 (other displacement data shows similar patterns), and the estimated ETF between the AC4–6 pair for experiment #4. The nearly zero region appears suddenly after 10 s, the time instant when the strong shaking
Figure 11. Results of experiment #3: (a) the acceleration record at AC7; (b) the acceleration record at AC4; (c) the displacement at DP14; (d) the estimated ETF for the AC4–7 pair (the horizontal axis is time); and (e) the rotated view of the same estimated ETF (the horizontal axis is frequency).

suddenly arrived, and develops rapidly. No obvious resonance mode is found after 10 s. Four large isolated spectral peaks are observed, and the one around 19 s correlates well with the time instant (18–20 s) during which sliding temporarily stops or slows down (see Figure 14(c)). It is not so clear from the recorded data if sliding also stops or slows down at the other three time instants, i.e. 13 s, 16 s, and 24 s. The same pattern is found for other trans-sliding-surface cases, i.e. the AC4–7 and AC4–8 pairs of experiment #4.

Similar to the cis-sliding-surface pairs of experiment #3, the estimated ETF for the AC6–7 pair (cis-sliding-surface) of experiment #4 shows no nearly zero region, and the same pattern is found for the other above-sliding-surface case, i.e. the AC6–8 pair of experiment #4.

Experiments #6. Experiment #6 represents a different in-situ condition since sand was added to the soft clay, and a weak sliding interface, i.e. the geo-synthetic interface, was inserted at the soft–stiff clay interface. It is found that the patterns in the estimated ETFs for the cis-sliding-surface cases of experiment #6 are quite different from those of previous experiments.

Figure 15 shows the profiles of the model slope of experiment #6, and Figure 16 shows the recorded data at AC4, AC9 (trans-sliding-surface) and DP17, and the estimated ETF between...
Figure 12. Results of experiment #3: (a) the acceleration record at AC7; (b) the acceleration record at AC6; (c) the displacement at DP14; (d) the estimated ETF for the AC6–7 pair (the horizontal axis is time); and (e) the rotated view of the same estimated ETF (the horizontal axis is frequency).

the AC4–9 pair for experiment #6. In this figure, we observed patterns similar to those seen from experiment #2, #3 and #4 (Figures 16(d) and (e)), e.g. the nearly zero region and the declination of the natural frequencies that do not return their original values at the end of the experiment. The identified natural frequency starts from roughly 15 Hz at 0 s, declines during 4–7 s, stays at roughly 4 Hz during 7–12 s, and returns to roughly 8 Hz after 12 s. Using the $V_s/4H$ rule, this mode implies that the shear wave velocity of the soft clay starts from 12.3 m/sec at 0 s, stays at 3.3 m/sec during 7–12 s, and returns to 7.0 m/sec after 12 s. Note that the degradation is more obvious than that which we have seen in Figure 16(e). The shear wave velocity of the soft clay evaluated by Wartman [43] using a small-strain test was 14.0 m/sec (as shown in Table II).

It is found that the estimated ETF for the AC7–9 pair (cis-sliding-surface) has the same patterns as Figures 16(d) and (e). This observation is completely different from the earlier situations: we have seen that for the previous experiments no nearly zero is present for all cis-sliding-surface cases. However, this is not the case for experiment #6, for which both AC7 and AC9 are above the weak interface, yet a nearly zero region is found.

The experimental observations obtained from experiment #6 are also quite different from those of the previous experiments. Wartman [43] reported that significant permanent
displacement of experiment #6 was caused by deviatoric straining of the sandy clay, while the permanent displacements of experiments #2, #3 and #4 were dominated by rigid body rotation along the deep circular sliding planes. Also, the observed acceleration data of experiment #6 is quite different. Basically, the input accelerations of experiments #3 and #6 are the same (see Figures 11(b) and 16(b)); however, the recorded acceleration on the top of the slope of experiment #6 (Figure 16(a)) is much smaller than that of experiment #3 (Figure 11(a)), indicating the energy was dissipated more rapidly for experiment #6.

**Remarks.** A common pattern in the estimated ETF for the trans-sliding-surface cases is a fully developed nearly zero region, which indicates the sliding of the model slopes. The durations during which these nearly zero regions are fully developed correlate very well with the durations of large permanent displacements of the model slopes. During these fully developed nearly zero regions, the high-frequency spectral ratios are small (usually less than 0.3), hence the ability of soil to transmit high-frequency energy between the any trans-sliding-surface pair of accelerometers declines significantly. Using the fully developed nearly zero region as an indicative, we are able to identify the initiating time and the end time of the major slope deformation.

When sliding occurs, there are time instants during which the sliding seems to stop or significantly slow down. These time instants can be identified in the recorded displacement time histories, i.e. 8 s and 10.5 s of experiment #3, and 19 s of experiment #4. At these time instants, the corresponding estimated ETFs show isolated high-frequency spectral peaks. Examples include 8 s and 10.5 s in Figures 11(d) and (e) and the four isolated peaks in Figures 14(d) and (e). The peaks indicate that the ability of the soil to transmit high-frequency energy is slightly improved during these time instants. However, detailed investigation
Figure 14. Results of experiment #4: (a) the acceleration record at AC6; (b) the acceleration record at AC4; (c) the displacement at DP16; (d) the estimated ETF for the AC4–6 pair (the horizontal axis is time); and (e) the rotated view of the same estimated ETF (the horizontal axis is frequency).

indicates that the high-frequency spectral ratios during these time instants are still nearly zero (see Figures 11(d) and (e) and Figures 14(d) and (e)). This observation suggests that when the sliding temporarily stops, high-frequency energy cannot freely propagate across the sliding plane.

Soil degradation, which manifests itself as decreases in natural frequencies, is observed in many cases. Also, in many cases we observe non-recoverable reduction in soil natural frequencies after strong excitations, e.g. the AC4–7 pair (Figures 11(d) and (e)). The degree of degradation and non-recoverable damage is more pronounced in the sandy clay case (experiment #6) than in the soft clay case (experiment #3). This observation is consistent with the conclusion of Ching and Glaser [33].

For experiment #3 (Figures 10(d) and (e) and 11(d) and (e)) and #6 (Figures 16(d) and (e)), we observed a gradual declination (to zero) of high-frequency spectral amplitudes prior to the fully developed nearly zero regions. We believe that this is indicative of minor damage prior to the primary failure. This pre-failure damage might indicate that the primary sliding planes did not instantaneously grow through the slopes for experiments #3 and #6. However, this pre-failure damage is not observed for experiment #4 (Figures 14(d) and (e)). This can be due to the fact that the strong motion arrived so suddenly for experiment #4 that the primary sliding plane formed instantaneously [43].
Figure 15. The pre-test and post-test profiles of experiment #6.

There is a major difference between the results of experiment #6 and those of experiments #2, #3 and #4: the estimated ETFs for the cis-sliding-surface cases have nearly zero regions for experiment #6. This indicates that the high-frequency energy cannot freely propagate in the soil above the geo-synthetic interface, but it freely propagates in the soils above the sliding planes of experiments #2, #3 and #4. We conclude that the soils above the deep sliding planes of experiments #2–4 should be relatively intact. However, the soil above the geosynthetic interface of experiment #6 should be damaged. This result agrees with the observations made by Wartman [43]: experiments #2, #3 and #4 are governed by deep circular/translational sliding planes, while experiment #6 is governed by distributed deviatoric straining and the sliding along the weak interface.

The non-parametric model with the thresholding estimator produced ETFs whose patterns are consistent with behaviors observed in the displacement data and the post-test slope profiles. The methodology has performed satisfactorily in identifying the rapidly time-varying soil slope systems.

7. DISCUSSION

We have seen from the two physical case studies that the wavelet non-parametric method produces meaningful results that are compatible with the conclusions of previous research (in the vertical seismic array examples) and experimental observations (in the seismic slope examples). We believe that the method has performed satisfactorily in these two case studies and is appropriate to analyze rapidly changing systems. The success of the wavelet non-parametric method in tracking rapidly changing systems is due to the ability of the NWT estimator in filtering rapidly changing functions without destroying their rapidly changing features, which is in turn due to the fact that wavelets are spatially adaptive and provide sparse representations.
Our approach has been successful in tracking the dynamical behaviors of the liquefying soil system at the Wildlife Refuge site in the sense that the approach is able to (a) identify the resonance frequency of the soil system before and after the initiation of liquefaction, (b) identify the initiation time of liquefaction, (c) find the time instants at which the soil significantly strain hardens, and (d) estimate the time instant after which the soil is nearly completely liquefied. The conclusions from our approach regarding these aspects are consistent with findings from previous research [9, 10]. Note that for items (c) and (d), previous research (i.e. Zeghal et al. [10]) computes the approximate stress–strain curves from the acceleration data. This approximation relies on the assumption that the 7.5 m thick soil layer deformed uniformly. Nevertheless, our approach uses an abstract mathematical model, which is free of physical assumptions, and found consistent conclusions with those of Zeghal et al. [10]. This justifies the results of Zeghal et al. and also indicates that our non-physical model is able to extract physical information from a physical process.

For the seismic slope experiment, we have seen that our model can successfully extract physical information from the data. We have also found indicatives for sliding and stopping of the slopes: a fully developed nearly zero region in the ETF implies sliding of the slopes, and an isolated high-frequency peak implies stopping of the slopes. These observations are useful for future research on seismic slope responses. Moreover, our results indicate that after the
slope slides, shear wave cannot freely propagate across the sliding plane because of the nearly zero regions, which in turns implies that ‘de-coupled’ analyses may not be applicable to this case study; otherwise, the high-frequency response of the slope may be over-estimated. That is, after the slope slides, we should consider the sliding plane as a discontinuous interface when modeling the slope. We have also seen that high-frequency energy can still not propagate across the sliding plane even when the slide temporarily stops or slows down.

8. CONCLUSIONS

A non-parametric statistical method based on wavelets is proposed to estimate the evolutionary transfer functions of rapidly changing dynamical systems. The method is a time-varying finite-impulse-response filter model. We have taken advantage of the spatial adaptation properties of the non-linear wavelet thresholding estimator to track rapidly changing systems, i.e. using the non-linear wavelet thresholding estimator, it is possible to estimate the evolutionary transfer function of a rapidly changing system without over-smoothing and under-smoothing the estimate. This cannot be achieved by linear estimators since they inherently assume the system changes uniformly with time. Two real data examples of rapidly changing systems are analyzed in this paper, including vertical seismic array case studies and seismic slope experiments. Most of our analysis results for the vertical seismic array data are consistent with previous research, while we have shown that our estimated evolutionary transfer is superior to those from previous research in the sense that our estimate is neither over-smoothed nor under-smoothed. For the seismic slope experiment data, our analysis results are also consistent with the experimental observations. Moreover, the proposed wavelet-based method is able to extract information that cannot be directly seen from the experimental data: for instance, the periods during which the slopes slide can be identified from the estimated evolutionary transfer function. From the estimated evolutionary transfer function, it is also concluded that after the slope slides, the shear wave cannot freely propagate across the sliding plane and we should consider the sliding plane as a discontinuous interface when modeling the slope, even when the slide stops or slows down.

APPENDIX A

In this Appendix, we explain the basic concept of the evolutionary power spectrum and show that our estimates for $a_p$ are consistent. To deal with a time-varying system, it is natural to consider non-stationary random processes. Indeed, a non-stationary process is closely related to a time-varying system, e.g. the output of a time-varying system is a non-stationary process. Priestley [44] defined a non-stationary process $X$ as

$$X(i) = \int_{-\pi}^{\pi} A_s(i, \omega) e^{j\omega i} d\xi(\omega) \quad i = 1, 2, \ldots, T$$  \hspace{1cm} (A1)$$

where $A_s(i, \omega)$ is a time-varying transfer function, and $\xi(\omega)$ is a random process with orthogonal increment in the frequency domain $[-\pi, \pi]$. Under this definition of a non-stationary process, a consistent estimate of $A_s(i, \omega)$ cannot be built from the observed process $X(1), \ldots, X(T)$. This is because the asymptotic considerations, e.g. consistency, efficiency, Central Limit
Theorem, etc., are meaningless: as \( T \) goes to \( \infty \), information about \( A_x(i, \omega) \) \((i \text{ finite})\) is not gained from the data infinitely away from time \( i \). Moreover, under the definition of Equation (A1), given a process \( X(1), \ldots, X(T) \), there are infinite corresponding \( A_x \) functions. This is because the number of relevant data points \( X \) around time \( i \) is always finite, and these finite data points can only uniquely define a local autocorrelation function of finite lags, while it generally requires a local autocorrelation function of infinite lags to uniquely define \( A_x(i, \omega) \).

In order to eliminate these drawbacks, Dahlhaus [29] defined a restrictive version of a non-stationary process—a locally stationary process—using the following definition: a sequence \( X \) is (zero-mean) locally stationary if there exists the following representation of \( X \),

\[
X(i/T) = \int_{-\pi}^{\pi} A_x(i/T, \omega) e^{i \omega \cdot \hat{x}(\omega)}
\]  

(A2)

where \( A_x(i/T, \omega) \) is a time-varying transfer function residing in \( H^1[0,1] \) in the time direction and in \( H^1[-\pi, \pi] \) in the frequency direction, \( \hat{x}(\omega) \) is a random process with orthogonal increment in the frequency domain \([-\pi, \pi]\). Under this definition, every function is defined on the finite closed interval \([0,1]\), and \( T \) extending to \( \infty \) no longer means the process extends to an infinite future but rather denser and denser information is observed in the interval \([0,1]\).

With this definition, the number of data points around time \( i/T \) will also increase to \( \infty \) as \( T \) goes to \( \infty \); therefore, it makes sense to consider the asymptotic properties of the estimate of \( A_x(i/T, \omega) \). For the same reason, we have to constrain the regularity of \( A_x(i/T, \omega) \) has too many discontinuities, the number of data points around time \( i/T \) that contains information about \( A_x(i/T, \omega) \) may not grow to \( \infty \) as \( T \) goes to \( \infty \). This is the reason \( A_x(i/T, \omega) \) is constrained as a function that is \( H^1[0,1] \) in the time direction and of \( H^1[-\pi, \pi] \) in the frequency direction. Dahlhaus also showed that \( A_x(i/T, \omega) \) is uniquely defined as \( T \) goes to \( \infty \) [45].

Dahlhaus further defined the so-called evolutionary spectrum \( f_{xx} \) of \( X \) as

\[
f_{xx}(i/T, \omega) = |A_x(i/T, \omega)|^2
\]  

(A3)

Under this definition, Dahlhaus [45] showed that the evolutionary spectrum of an ARMA process with time-varying coefficients, i.e. the \( X \) in the following equation:

\[
\sum_{m=0}^{a} a_m(i/T) \cdot X[(i - m)/T] = \sum_{m=0}^{b} b_m(i/T) \cdot \varepsilon[(i - m)/T]
\]  

(A4)

where \( \varepsilon(i/T) \) is independent noise with variance \( \sigma^2(i/T) \), is exactly

\[
f_{xx}(i/T, \omega) = \frac{\sigma(i/T)^2 \cdot |\sum_{m=0}^{b} b_m(i/T) e^{i\omega m}|^2}{2\pi \cdot |\sum_{m=0}^{a} a_m(i/T) e^{i\omega m}|^2}
\]  

(A5)

Now consider another locally stationary process \( Y \). We can similarly define the evolutionary cross spectrum between \( X \) and \( Y \) as \( f_{xy} \),

\[
f_{xy}(i/T, \omega) = A_x(i/T, \omega)^* \cdot A_y(i/T, \omega)
\]  

(A6)

We can apply Equation (14) to estimate \( f_{xx} \) and \( f_{xy} \), and we state without proof some statistical properties of the estimates \( \hat{f}_{xx} \) and \( \hat{f}_{xy} \). The theorem proven by von Sachs and
Schneider [46] states that (1) if $N$ is asymptotically less than $T^{1/2}/\ln(T)$ and larger than $T^{1/4}$ and (2) $A_x(t, \omega)$ is of $H^1[0,1]$ in $t$ and $\omega$, $\hat{f}_{xx}$ is a consistent estimator of $f_{xx}$. In addition, they derived an asymptotic normal distribution for the wavelet coefficients of $f_{xx}$. The same theorem also applies to $f_{xy}$. The Heisenberg Uncertainty Principle [21] states that one cannot obtain arbitrarily good resolution in both the time and frequency domain. The selection window width $N$ is exactly a tradeoff between the time-domain and frequency-domain resolution. A small $N$ makes the time-domain resolution better but worsens the frequency resolution. The asymptotic bound between $T^{1/4}$ and $T^{1/2}/\ln(T)$ is the balance region of this tradeoff, i.e. the lower bound $T^{1/4}$ and the upper bound $T^{1/2}/\ln(T)$ insure that the frequency-domain and time-domain biases of $\hat{a}_p$, respectively, decrease with $T$. Also note that $N$ grows with $T$, but grows slower than $T$. This is required for the approach to be non-parametric.

Given a linear time-invariant system, its transfer function can be estimated as the ratio of the cross spectrum between input and output and the power spectrum of the input. Similarly, for the time-varying system of Equation (11), the ETF is expected to be:

$$B(i/T, \omega) \simeq \frac{f_{xy}(i/T, \omega)}{f_{xx}(i/T, \omega)}$$ (A7)

Indeed, Chiann and Morettin [47] showed that under the assumption that $N$ is asymptotically less than $T^{1/2}/\ln(T)$ and larger than $T^{1/4}$,

$$f_{xy}(i/T, \omega) = B(i/T, \omega)f_{xx}(i/T, \omega) + O(T^{-1})$$ (A8)

provided that $e(i/T)$ of Equation (11) is independent of $X((i - m)/T)$ for $m > 0$. After $\hat{f}_{xx}$ and $\hat{f}_{xy}$ are obtained, we can find $\hat{a}_p$ using Equation (15). The theorem proven by Chiann and Morettin states that $\hat{a}_p$ is also a consistent estimator of $a_p$, but the asymptotic distribution and convergence rate of $\hat{a}_p$ are not derived.

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